Interaction Information: Linear and Nonlinear Interpretations

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In his article entitled "Ross Ashby's information theory: a bit of history, some solutions to problems, and what we face today," Krippendorff (2009) argues against interaction information among three (or more) variables Q as a viable measure in information theory (McGill, 1954; Yeung, 2008, at pp. 51 ff.). Following Ashby's (1969) definitions, the author defines transmission as the difference between the sum of the entropies for variables independently and the uncertainty in the system of all variables taken together (at p. 192):

$$T(\mathrm{A{:}B}) = H(\mathrm{A}) + H(\mathrm{B}) - H(\mathrm{AB}), \tag{1}$$
$$T_\mathrm{C}(\mathrm{A{:}B}) = H_\mathrm{C}(\mathrm{A}) + H_\mathrm{C}(\mathrm{B}) - H_\mathrm{C}(\mathrm{AB}), \tag{2}$$
$$T(\mathrm{A{:}B{:}C}) = H(\mathrm{A}) + H(\mathrm{B}) + H(\mathrm{C}) - H(\mathrm{ABC}). \tag{3}$$

The amount of interaction involving three variables $Q(\mathrm{ABC})$ can then be formulated as:

$$Q(\mathrm{ABC}) = T_\mathrm{C}(\mathrm{A{:}B}) - T(\mathrm{A{:}B}) = T_\mathrm{B}(\mathrm{A{:}C}) - T(\mathrm{A{:}C}) = T_\mathrm{A}(\mathrm{B{:}C}) - T(\mathrm{B{:}C}), \tag{4}$$
$$T(\mathrm{A{:}B{:}C}) = T(\mathrm{A{:}B}) + T(\mathrm{A{:}C}) + T(\mathrm{B{:}C}) + Q(\mathrm{ABC}) \tag{5}$$

Ashby (like many others) wished to explain this Q-measure as the amount of information due to the unique combination of a number of variables which was not reducible to any of its subsets. Krippendorff (2009, at pp. 199f.) shows that this logic is "faulty" (at p. 196) because $Q(\mathrm{ABC})$ cannot be considered as information. In order to show this, Krippendorff (at p. 199) rewrites equations 4 and 5 as follows:[1]

$$Q(\mathrm{A;B}) = Q_\mathrm{B}(\mathrm{A}) - Q(\mathrm{A}) = T(\mathrm{A;B}) \tag{6}$$

$$Q(\mathrm{A;\,B;\,C}) = Q_\mathrm{C}(\mathrm{A;B}) - Q(\mathrm{A;B}) \tag{7}$$

$$= \sum_{abc} P(a,b,c)\log\frac{P(a,b\,|\,c)}{P(a\,|\,c)P(b\,|\,c)} - \sum_{AB} P(a,b)\log\frac{P(a,b)}{P(a)P(b)}$$

[…]

$$= \sum_{abc} P(a,b,c)\log\frac{P(a,b,c)}{\left[\dfrac{P(a,b)P(a,c)P(b,c)}{P(a)P(b)P(c)}\right]} \tag{8}$$

[1] For reasons of parsimony, I use the three-way interaction, but the reasoning for higher orders is analogous.

Equation 8 might be considered as a Kullback-Leibler divergence in which $\Sigma_{abc}\, P_{abc}$ would be the composed or posterior probability distribution, and the denominator would represent the prior or composing elements. However, Krippendorff (2009, at p. 200) noted that the denominator of this equation no longer sums to 1, and that therefore this cannot be considered as a probability distribution and hence can no longer be expressed within the framework of information theory (cf. Yeung, 2008, at pp. 59f.). *Q*(ABC) is to be discarded as incompatible with information theory: *circular* relationships among the components are introduced, whereas Shannon's theory assumes only *linear* relationships. From the linear perspective, "a message received could have no effect on the message sent" (p. 197).

McGill (1954) noted that from the perspective of statistics *Q* measures associations between variables, and not the direction of the transmission: "This means that nothing is gained formally by distinguishing transmitters from receivers" (at p. 100). The formal approach of statistics thus leaves the Shannon framework of linear transmissions behind, and then provides us with a *Q*-measure in the case of more than two variables, which can also be negative. Following Krippendorff, these formal results would need an interpretation different from the one provided by Ashby (1969), Krippendorff (1979), and others working within the "linear" program. The problem is, in my opinion, not the measure, but the theoretical appreciation. Furthermore, Garner and McGill (1956) showed that the information-theoretical approach is identical to the multivariate analysis of variance, except that "uncertainty analysis has the greater generality and requires no assumptions about metric properties of the criterion" (p. 227). The *Q*-measure is mathematically sound, but an interpretation from the perspective of information theory still needs to be provided.

Theil (1972, at p. 13) argued for two different interpretations of the entropy concept. A second interpretation "goes far beyond the narrow area of messages and even that of probabilities." According to Theil, uncertainty prior to the arrival of the message and expected information provided by the message can be considered as two sides of the same coin. The latter can be measured as a Kullback-Leibler divergence, while the former can be measured in terms of the expected information content of a system of variables at each moment of time. Theil (1972) uses the expected information content for the statistical decomposition at a specific moment of time, and the Kullback-Leibler divergence for the dynamic extension. The grouping variable in a statistical decomposition, for example, can be considered as conditioning the grouped variable and transmission can be discussed in this (static) framework as a measure of associations (Leydesdorff, 1991). The difference between prior and posterior probability distributions, however, can be studied as a Kullback-Leibler divergence. In this latter case, the information is necessarily non-negative (Theil, 1972, pp. 59f.).

As noted, *Q*(AB) = *T*(AB) in the case of two variables. However, in the case of three (or more) variables *Q* can be negative, while *T* (as defined in Eq. 3) is necessarily non-negative. Jakulin (2005; cf. Jakulin & Bratko, 2004) explained this possibly negative value of *Q* by distinguishing between synergy and redundancy in the case of a three-way interaction: "The notion that the same information comes from two sources, and to have this represented as a negative number that serves to correct for overlap, is very elegant. It is measured in bits, it serves a particular need, and for all point-to-point communication purposes, it is zero or positive" (Jakulin, personal communication, March 1, 2009). In the opposite case of synergy, the third system C establishes a connection between A and B which does not overlap with the mutual information between A and B. The configuration generates a path for the information transfer between A and B, in addition to the mutual information I_{AB}. Q_{ABC} is in this case positive.

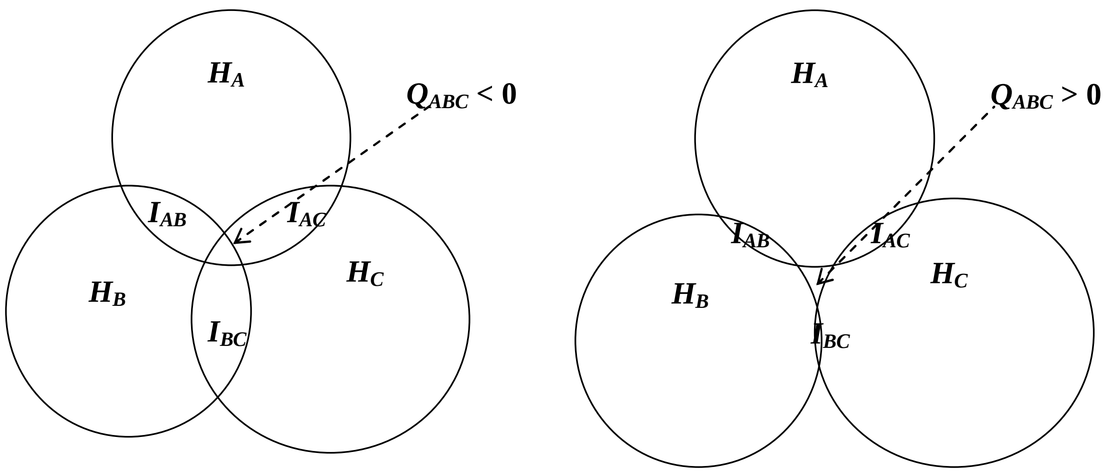

Figure 1: Relations between probabilistic entropies (*H*), transmissions (*I*), and configurational information (*Q*) for three interacting variables *A, B,* and *C*.

Figure 1 shows the two configurations. The relative weights of and overlaps between the sets determine whether Q_{ABC} is positive or negative. This is expressed in Abramson's (1963, at p. 129) formulation of the mutual information among three alphabets as follows:

$$I(\text{ABC}) = H(\text{A}) + H(\text{B}) + H(\text{C}) - H(\text{AB}) - H(\text{AC}) - H(\text{BC}) + H(\text{ABC})$$

This measure is similar to Q(ABC), but with the opposite sign. Because of the different sign, I(ABC) measures uncertainty (with a positive sign) or uncertainty reduction (with a negative sign) in the configuration among three (or more) variables or subsystems. I have used this measure, for example, as an indicator of the positive or negative feedback in the interactions in a Triple Helix of university-industry-government relations (Leydesdorff, 2006, 2008).

Beyond this application, the third dimension C can also be considered as a positive or negative feedback loop added to the linear flux between A and B. Maturana (e.g., 2000) explained *autopoiesis* as "the coordination of coordinations in a recursive process that operates on a linear flow." The linear flow (between A and B) develops historically with the arrow of time and generates (necessarily) positive entropy. Recursive feedback operates against the arrow of time and can potentially generate redundancy, albeit locally, since contained within a circular loop. Both dynamics operate concurrently: the outcome of this process between self-organization as a coordinated coordination versus the historical development of the coordinations along the axis of time can be measured using the *Q*-measure. The uncertainty at the systems level is reduced if the configuration provides a synergy. Note that Maturana's "coordination of coordinations" cannot be measured in terms of information without the specification of *another* dimension of the communication. In addition to information, meaning is generated and potentially communicated in the loop (Leydesdorff & Franse, 2009).

In the case of the Triple Helix dynamics of university-industry-government relations I_{ABC} (= - Q_{ABC}) provides us with an indicator of an innovation system as a configuration in which uncertainty can be reduced because of an emerging circularity at the systems level. The consequent reduction of uncertainty cannot be attributed to one of the composing elements or to their mutual relations. In the opposite case, government, for example, may be able to direct both the economy and science in a hierarchical model and redundancy would be generated in the relations (Park & Leydesdorff, in preparation). Using a further extension to four dimensions, Leydesdorff & Sun (2009) found an erosion of the national system in the case of Japan which was counteracted by an emerging system that includes international relations. Sun & Negishi (2008) have shown recently that similar results could be obtained using partial correlation analysis.

This configurational or interaction information enables us to measure the imprint of a recursive loop on the linear flux (Lucio-Arias & Leydesdorff, 2009). As Krippendorff (2009) observed, this measurement of a circular relationship can be considered as theoretically incommensurable with Shannon's and Ashby's program of linear relationships between a sender and a receiver. However, it does not conflict with a statistical approach that uses information theory for the measurement of specific associations in a non-linear dynamics (Brooks & Wiley, 1986; Swenson, 1989).

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